Robustness and surgery of frames

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1. Introduction

The focus of this paper is on robustness and surgeries of frames in finite dimensional Hilbert spaces. The study of finite dimensional frames has been motivated by a variety of applications such as signal processing, multiple-antenna wireless systems, and sampling theory. The concept of frames was introduced by Duffin and Schaeffer [5] and popularized by Daubechies [4]. A good introduction to frames in finite dimensional Hilbert spaces can be found in [7].

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A frame in a finite dimensional Hilbert space is a redundant spanning set of vectors. We consider two operations on frames. The \textit{erasure} consists of removing vectors in a frame. A frame is said to be \textit{robust} to \( k \) erasures if after randomly removing \( k \) vectors the resulting set is still a frame. In this paper we give a necessary and sufficient condition for a frame in \( \mathbb{R}^n \) to be robust to \( k \) erasures. The condition is stated in terms of the support of the null space of the synthesis operator of the frame. This theorem generalizes a characterization of frames robust to one erasure given by Casazza and Kovačević [2]. The second operation that we consider is the removal of \( r \) vectors from the frame and adding \( k \) vectors to the frame called \((r, k)\)-surgery. We characterize when a \((r, k)\)-surgery is possible on a unit-norm tight frame in \( \mathbb{R}^2 \). The result on “length surgery” generalizes a characterization of the existence of tight frames with prescribed norms found in [3].

2. Preliminaries

We begin with the definition of a frame.

\textbf{Definition 2.1.} A \textit{frame} for a Hilbert space \( \mathcal{H} \) is a sequence of vectors \( \{x_i\}_{i \in I} \subset \mathcal{H} \) for which there exist constants \( 0 < A \leq B < \infty \) such that for every \( x \in \mathcal{H} \),

\[ A\|x\|^2 \leq \sum_{i \in I} |\langle x, x_i \rangle|^2 \leq B\|x\|^2. \]

Here \( A \) is the greatest lower frame bound and \( B \) is the least upper frame bound. A frame is called a \textit{tight frame} if \( A = B \). A \textit{uniform frame} is a frame in which all the vectors have equal norm. If all the norms in a uniform frame equals one, the frame is called a \textit{unit-norm frame}. We focus on frames in finite dimensional Hilbert spaces.

Let \( \{x_i\}_{i \in I} \) be a frame in \( \mathcal{H} \). The linear map \( V : \mathcal{H} \rightarrow l^2(I) \) defined by \( (Vx)_i = \langle x, x_i \rangle \) is called the \textit{analysis operator}. The Hilbert space adjoint \( V^* \) is called the \textit{synthesis operator}. The \textit{frame operator} is given by \( V^*V \). In \( \mathbb{R}^n \) the analysis operator of a frame \( \{x_i\}_{i=1}^m \) is given by the \( m \)-by-\( n \) matrix

\[
V = \begin{bmatrix}
\leftarrow x_1^* \rightarrow \\
\leftarrow x_2^* \rightarrow \\
\vdots \\
\leftarrow x_m^* \rightarrow 
\end{bmatrix}
\]

and the \textit{synthesis operator} \( V^* \) is given by the \( n \)-by-\( m \) matrix

\[
V^* = \begin{bmatrix}
\uparrow \uparrow \uparrow \\
x_1 & x_2 & \ldots & x_m \\
\downarrow \downarrow \downarrow 
\end{bmatrix}
\]

and the \textit{frame operator} is the matrix \( V^*V \).

The following equivalent descriptions of frames in \( \mathbb{R}^n \) are used in this paper.

\textbf{Theorem 2.2} [7]. The following statements are equivalent:

1. \( \{x_1, x_2, \ldots, x_m\} \) is a frame in \( \mathbb{R}^n \).
2. \( \{x_1, x_2, \ldots, x_m\} \) is a spanning set for \( \mathbb{R}^n \).
3. The \( n \)-by-\( m \) matrix \( V^* = \begin{bmatrix} x_1 & x_2 & \ldots & x_m \end{bmatrix} \) has rank \( n \).
3. Robustness

Suppose an encoded version $Vx$ of a vector $x$ in $\mathbb{R}^n$ is transmitted across a communication network. If $k$ of the components of $Vx$ are lost or not delivered the receiver would want to be able to reconstruct $Vx$ using the components that have been received. If $V$ represents the analysis operator of a frame $\{x_i\}_{i=1}^m$ in $\mathbb{R}^n$, what are the frames that allow one to recover the coefficients corresponding to the $k$ frame vectors that are “erased” during transmission? Such frames are often called robust to $k$ erasures. Notice that these are exactly the frames that remain frames after any $k$ frame vectors are “erased” during transmission. The characterization shows that every index set of size $m - k + 1$ is in the support of the null space of the synthesis operator.

**Definition 3.2.** The support of a vector $x = (x_1, x_2, \ldots, x_m)$ in $\mathbb{R}^m$, denoted $\text{supp}(x)$, is the set of indices $\{i \in \{1, 2, \ldots, m\} : x_i \neq 0\}$.

**Definition 3.3.** Let $\mathcal{N}(A)$ denote the null space of a matrix $A$. The support of the null space of $A$, denoted $\nabla(A)$, is the collection of $\text{supp}(x)$ as $x$ varies over $\mathcal{N}(A)$. That is,

$$\nabla(A) = \{\text{supp}(x) : x \in \mathcal{N}(A)\}.$$ 

**Example 3.4.** Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$ 

Then $\mathcal{N}(A) = \text{span}\{v_1, v_2\}$ where $v_1^T = (1, 0, -1, 1, -1)$ and $v_2^T = (0, 1, -1, 0, 1)$. To determine $\nabla(A)$ we consider $\text{supp}(x)$ where $x = \alpha v_1 + \beta v_2, \alpha, \beta \in \mathbb{R}$.

**Case 1:** if $\alpha = 0$, then the possibilities for $\text{supp}(x)$ are $\emptyset$ or $\{2, 3, 5\}$. 

**Case 2:** if $\beta = 0$, then the possibilities for $\text{supp}(x)$ are $\emptyset$ or $\{1, 3, 4, 5\}$. 

**Case 3:** if $\alpha \neq 0$ and $\beta \neq 0$ then the possibilities for $\text{supp}(x)$ are $\{1, 2, 3, 4, 5\}$, or $\{1, 2, 4, 5\}$. 

Hence

$$\nabla(A) = \{\{2, 3, 5\}, \{1, 3, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{1, 2, 3, 4, 5\}\}.$$ 

**Lemma 3.5.** If $X, Y \in \nabla(A)$, then $X \cup Y \in \nabla(A)$. 

**Proof.** Suppose $x, y \in \mathcal{N}(A)$ such that $X = \text{supp}(x)$ and $Y = \text{supp}(y)$. For any $\epsilon \in \mathbb{R}$, let $z_\epsilon = x + \epsilon y$. Clearly $z_\epsilon \in \mathcal{N}(A)$ and as the $i$th component of $z_\epsilon$ is $x_i + \epsilon y_i$, we see that for all but a finite number of $\epsilon$ we have $\text{supp}(z_\epsilon) = X \cup Y$. Thus $X \cup Y \in \nabla(A)$.  

The following classification of frames robust to one erasure was given by Casazza and Kovačević.
Theorem 3.6 [3]. Let \( \{x_i\}_{i=1}^{m} \) be a frame in \( \mathbb{R}^n \). The following are equivalent:

1. \( \{x_i\}_{i=1}^{m} \) is a frame robust to one erasure.
2. There are scalars \( c_i \neq 0 \), for \( 1 \leq i \leq m \), such that
   \[
   \sum_{i=1}^{m} c_i x_i = 0.
   \]

Remark 3.7. Statement (2) is a condition on the support of the null space of the synthesis operator, in particular: \( \{1, 2, \ldots, m\} \in \overset{\sim}{\nabla}(V^*) \), where \( V^* \) is the synthesis operator corresponding to the frame \( \{x_i\}_{i=1}^{m} \). This observation motivated the following theorem.

Theorem 3.8. Let \( \{x_i\}_{i=1}^{m} \) be a frame in \( \mathbb{R}^n \). The following are equivalent:

1. \( \{x_i\}_{i=1}^{m} \) is a frame robust to \( k \) erasures.
2. For all index sets \( I \subset \{1, 2, \ldots, m\} \) with \( |I| = k - 1 \), \( I^c \in \overset{\sim}{\nabla}(V^*) \).

Here if \( I \subset \{1, \ldots, m\} \) we let \( I^c \) denote the complement of \( I \), that is \( I^c = \{1, \ldots, m\} \setminus I \).

Proof. (2) \( \Rightarrow \) (1) Suppose that \( x_{j_1}, x_{j_2}, \ldots, x_{j_k} \) are erased from \( \{x_i\}_{i=1}^{m} \). We will first show that \( x_{jk} \) can be reconstructed from the remaining frame vectors after erasing \( x_{j_1}, \ldots, x_{j_k} \). By hypothesis, \( J = \{j_1, \ldots, j_{k-1}\}^c \in \overset{\sim}{\nabla}(V^*) \) so there exists a vector \( c \in \mathcal{N}(V^*) \) such that \( \text{supp}(c) = J \). Therefore

\[
0 = V^*c = \sum_{i \in J} c_i x_i = c_k x_{jk} + \sum_{i \in J, i \neq j_k} c_i x_i.
\]

Since \( c_{jk} \neq 0 \) we obtain

\[
x_{jk} = \sum_{i \in J, i \neq j_k} \left( -\frac{c_i}{c_{jk}} \right) x_i.
\]

In a similar fashion each of \( x_{j_1}, \ldots, x_{j_{k-1}} \) can be also be reconstructed from the frame vectors left after erasing \( x_{j_1}, \ldots, x_{j_k} \).

Finally, as \( \{x_1, \ldots, x_m\} \) span \( \mathbb{R}^n \) and the span of \( \{x_i\}_{i \in J} \) includes the vectors \( \{x_i\}_{i \in I^c} \) we must have that \( \{x_i\}_{i \in J} \) is a spanning set and hence a frame.

(1) \( \Rightarrow \) (2) Suppose \( \{x_i\}_{i=1}^{m} \) is a frame robust to \( k \) erasures. Let \( I = \{j_1, j_2, \ldots, j_{k-1}\} \subset \{1, 2, \ldots, m\} \). Let \( J \in \overset{\sim}{\nabla}(V^*) \) be an index set of maximum cardinality disjoint from \( I \). We claim that \( J = I^c \). Suppose not, then \( J \subsetneq I^c \) and we may choose \( j_k \in J \cap I^c \). Since \( \{x_i\}_{i=1}^{m} \) is robust to \( k \) erasures, \( x_{jk} \) can be reconstructed from frame vectors remaining after the erasure of \( x_{j_1}, \ldots, x_{j_k} \). Thus

\[
x_{jk} = \sum_{i \in I^c, i \neq j_k} c_i x_i.
\]

and so

\[
V^*y = x_{jk} - \sum_{i \in I^c, i \neq j_k} c_i x_i = 0
\]

where \( y \) is the vector with components \( y_i = -c_i \) when \( i \in I^c \) and not equal to \( j_k \), \( y_{jk} = 1 \), and \( y_i = 0 \) when \( i \notin I \). Then \( \text{supp}(y) \in \overset{\sim}{\nabla}(V^*) \), \( J_k \in \text{supp}(y) \), and \( \text{supp}(y) \subset I^c \). By Lemma 3.5, we also have that \( J \cup \text{supp}(y) \in \overset{\sim}{\nabla}(V^*) \). This contradicts the assumption that \( J \) has maximum cardinality. Hence \( J = I^c \in \overset{\sim}{\nabla}(V^*) \). \( \square \)
Example 3.9. Suppose
\[
A = \begin{bmatrix}
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0
\end{bmatrix}.
\]
Using Theorem 2.2 we observe that the columns of \(A\) form a frame in \(\mathbb{R}^3\). In Example 3.4 we found that
\[
\tilde{\nabla}(A) = \{ \{2, 3, 5\}, \{1, 3, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{1, 2, 3, 4, 5\} \}.
\]
Notice that \(\{1\} = \{2, 3, 4, 5\} \not\in \tilde{\nabla}(A)\). Hence from Theorem 3.8, the columns of \(A\) do not form a frame robust to 2 erasures.

4. Frame surgery

Given a frame, a natural question is whether it can be manipulated in some way to make it a tight frame. Recall that a tight frame is a frame with equal upper and lower bounds. One kind of manipulation of frames in \(\mathbb{R}^n\) is to maintain the lengths of vectors but change the orientation of the vectors. Another kind of manipulation is to add or remove some vectors from a given frame.

Definition 4.1 [7]. An \((r, k)\)-surgery on a finite sequence of vectors in \(\mathbb{R}^n\) removes \(r\) vectors and adds \(k\) vectors to the sequence.

A frame \(\{x_i\}_{i=1}^m\) in \(\mathbb{R}^2\) can be represented in polar coordinates as
\[
x_i = \begin{bmatrix}
a_i \cos \theta_i \\
a_i \sin \theta_i
\end{bmatrix},
\]
where \(a_i\) is the length of \(x_i\) and \(0 \leq \theta_i \leq \pi\) is the angle between \(x_i\) and the positive \(x\)-axis. Suppose \(\{x_i\}_{i=1}^m\) is a tight frame. Then \(V^*V = al\) where \(a\) is the frame bound and \(I\) is the identity matrix. Thus
\[
V^*V = \begin{bmatrix}
\sum a_i^2 \cos^2 \theta_i & \sum a_i^2 \cos \theta_i \sin \theta_i \\
\sum a_i^2 \cos \theta_i \sin \theta_i & \sum a_i^2 \sin^2 \theta_i
\end{bmatrix} = \begin{bmatrix}
a & 0 \\
0 & a
\end{bmatrix}
\]
is bounded by \(\frac{1}{2}(k^2 - N)\) where \(N = m - r\) and \(\tilde{x}_1, \ldots, \tilde{x}_N\) denote the diagram vectors that remain after deleting \(r\) frame vectors.
Proof. Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N$ be the diagram vectors that are present after removing $r$ vectors from the unit-norm tight frame $\mathcal{F}$. Let $W$ denote their vector sum $\vec{x}_1 + \cdots + \vec{x}_N$. If we were to add $k$ unit vectors to $\mathcal{F}$ and keep it a unit-norm tight frame then such a $(r, k)$-surgery on $\mathcal{F}$ is possible if and only if $\|W\| \leq k$. Therefore,

$$\|W\|^2 = \langle \vec{x}_1 + \cdots + \vec{x}_N, \vec{x}_1 + \cdots + \vec{x}_N \rangle$$

$$= \sum_{i=1}^{N} \|\vec{x}_i\|^2 + \sum_{i,j=1}^{N} \langle \vec{x}_i, \vec{x}_j \rangle$$

$$= N + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \langle \vec{x}_i, \vec{x}_j \rangle$$

$$= N + 2S$$

where $S$ is the sum of the strict upper triangular part of the Grammian $\{(\vec{x}_i, \vec{x}_j)\}_{i,j=1}^N$. Hence $(r, k)$-surgery is possible if and only if

$$N + 2S \leq k^2$$

or equivalently

$$S \leq \frac{k^2 - N}{2}. \quad \square$$

Remark 4.4. If $\theta_{ij}$ denotes the angle between vectors $\vec{x}_i$ and $\vec{x}_j$ then from Theorem 4.3 we get

$$S = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \cos (\theta_{ij}) \leq \frac{k^2 - N}{2}. \quad (4.4)$$

Corollary 4.5. Let $\{x_i\}_{i=1}^m$ be a unit-norm tight frame in $\mathbb{R}^2$. Suppose $k = \lfloor \frac{m}{2} \rfloor$ where $m > 2$. Then it is always possible to perform a $(k + 1, k)$-surgery resulting in a new unit-norm tight frame. In particular, any unit-norm tight frame consisting of $m$ vectors may be reduced to a unit-norm tight frame consisting of $m - s$ vectors where $1 \leq s \leq m - 2$.

Proof. Suppose $m = 2k + 1$. Then removing $k + 1$ vectors leaves $N = m - (k + 1) = k$ vectors. Because $\cos (\theta_{ij}) \leq 1$, using Remark 4.4, we get

$$S = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \cos (\theta_{ij}) \leq \frac{k(k - 1)}{2} = \frac{k^2 - N}{2}. \quad (4.5)$$

Hence from Theorem 4.3 it is possible to perform $(k + 1, k)$-surgery.

Suppose $m = 2k$. Then removing $k + 1$ vectors leaves $N = m - (k + 1) = k - 1$ vectors. Again

$$S = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \cos (\theta_{ij}) \leq \frac{k(k - 1)}{2} < \frac{k^2 - (k - 1)}{2} \quad (4.6)$$

implies from Theorem 4.3 that $(k + 1, k)$-surgery is possible.

It is easy to observe that a repeated application of the two cases in the proof will reduce any unit-norm tight frame of $\mathbb{R}^2$ consisting of $m$ vectors to a unit-norm tight frame of $\mathbb{R}^2$ consisting of $m - s$ vectors where $1 \leq s \leq m - 2. \quad \square$

Remark 4.6. A construction of the vectors for Corollary 4.5 in the case $m = 2k + 1$ can be shown as follows.

For $\theta \in [0, 2\pi)$, let $R_\theta$ be the matrix that rotates a vector $x$ in $\mathbb{R}^2$ by $\theta$ radians in the counterclockwise direction. That is,
Let $\{x_i\}_{i=1}^m$ be a unit-norm tight frame for $\mathbb{R}^2$ where $m = 2k + 1$. Suppose the vectors $x_1, \ldots, x_{k+1}$ are removed from $\{x_i\}_{i=1}^m$ and assume that the remaining vectors are not all identical. Define $a, b \in \mathbb{R}$ as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \sum_{i=k+2}^m \bar{x}_i.$$ 

To simplify calculations, pick the angle $\theta$ such that

$$R_{\theta} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sqrt{a^2 + b^2} \\ 0 \end{bmatrix} = \sum_{i=k+2}^m R_{\theta} \bar{x}_i.$$ 

Note that $\{R_{\theta} \bar{x}_i\}_{i=k+2}^m$ are the diagram vectors of $\{R_{\theta} x_i\}_{i=1}^{m-1}$. Define $\phi_i \in [0, 2\pi)$ for $k+2 \leq i \leq m$, so that

$$R_{\theta} x_i = \begin{bmatrix} \cos \phi_i \\ \sin \phi_i \end{bmatrix}$$

and

$$R_{2\theta} \bar{x}_i = \begin{bmatrix} \cos 2\phi_i \\ \sin 2\phi_i \end{bmatrix}.$$ 

Consider the vectors $\{y_i\}_{i=1}^k$ where

$$y_i = \begin{bmatrix} \cos \left(\frac{\pi}{2} - \phi_{k+1+i} \right) \\ \sin \left(\frac{\pi}{2} - \phi_{k+1+i} \right) \end{bmatrix} = \begin{bmatrix} \sin(\phi_{k+1+i}) \\ \cos(\phi_{k+1+i}) \end{bmatrix}.$$ 

Then for $1 \leq i \leq k$,

$$\bar{y}_i = \begin{bmatrix} \cos(\pi - 2\phi_{k+1+i}) \\ \sin(\pi - 2\phi_{k+1+i}) \end{bmatrix} = \begin{bmatrix} -\cos 2\phi_{k+1+i} \\ \sin 2\phi_{k+1+i} \end{bmatrix}.$$ 

Consider the sequence $\mathcal{F} = \{R_{\theta} x_i\}_{i=k+2}^m \cup \{y_i\}_{i=1}^k$. Clearly $R_{\theta} x_i$ and $y_j$ are linearly independent when $\cos \phi_i \neq \sin \phi_i$. As $\{x_i\}_{i=k+2}^m$ are all not identical there must exist vectors $R_{\theta} x_i$ and $y_j$ that are linearly independent. So $\mathcal{F}$ spans $\mathbb{R}^2$ and therefore is a frame. Also,

$$\sum_{i=k+2}^m R_{2\theta} \bar{x}_i + \sum_{i=1}^k \bar{y}_i = \sum_{i=k+2}^m \begin{bmatrix} \cos 2\phi_i \\ \sin 2\phi_i \end{bmatrix} + \sum_{i=1}^k \begin{bmatrix} -\cos 2\phi_{k+1+i} \\ \sin 2\phi_{k+1+i} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{a^2 + b^2} \\ 0 \end{bmatrix} + \begin{bmatrix} -\sqrt{a^2 + b^2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

shows that $\mathcal{F}$ is a tight frame.

When considering tight frames, a natural question arises: when do the norms of vectors in a frame automatically prohibit the frame from being a tight frame? The question is answered in [2] for an $n$-dimensional Hilbert space.
Theorem 4.7 [2]. There is a tight frame for an n-dimensional Hilbert space $\mathcal{H}$ with $m$ vectors having norms $\|x_i\| = a_i$, $i = 1, 2, \ldots, m$ if and only if the following inequality is satisfied:

$$\max_{1 \leq i \leq m} \{a_i^2\} \leq \frac{1}{n} \sum_{i=1}^{m} a_i^2.$$

Definition 4.8. Let $a_1, a_2, \ldots, a_m$ denote norms of $m$ vectors in $\mathbb{R}^n$. An $(r, k)$-length surgery on $\{a_i\}_{i=1}^{m}$ removes $r$ numbers and replaces them with $k$ positive numbers.

We have the following generalization of Theorem 4.7.

Theorem 4.9. Given a sequence $\{a_i\}_{i=1}^{m}$ of nonnegative real numbers, it is possible to perform a $(0, k)$-length surgery resulting in a sequence of nonnegative real numbers that corresponds to norms of vectors in a tight frame for $\mathbb{R}^n$ if and only if $0 \leq k < n$ and

$$a_{\text{max}}^2 = \max_{1 \leq i \leq m} \{a_i^2\} \leq \frac{1}{n-k} \sum_{i=1}^{m} a_i^2.$$

Proof. Let $M = \sum_{i=1}^{m} a_i^2$ and consider $\{r_i\}_{i=1}^{k} \cup \{a_i\}_{i=1}^{m}$. If $a_{\text{max}}^2 \leq \frac{1}{n} M$, then the fundamental inequality holds for all $k \in \mathbb{N}$ if $r_1 = 0$ for all $1 \leq i \leq k$. Otherwise, $a_{\text{max}}^2 > \frac{1}{n} M$. With the addition of $\{r_i\}_{i=1}^{k}$ the left hand side of the fundamental inequality becomes:

$$\max\{a_{\text{max}}^2, r_1^2, r_2^2, \ldots, r_k^2\}$$

while the right hand side becomes:

$$\frac{1}{n} (M + r_1^2 + r_2^2 + \cdots + r_k^2).$$

As we want Eq. (1) to be as small as possible and Eq. (2) to be as large as possible, we may assume $r := r_1 = \cdots = r_k$. Then Eq. (1) becomes $\max\{a_{\text{max}}^2, r^2\}$ while Eq. (2) becomes $\frac{1}{n} (M + k r^2)$. Define the function $h : [0, \infty) \rightarrow \mathbb{R}$ as

$$h(r) = \begin{cases} \frac{(M + kr^2)}{n} - a_{\text{max}}^2 & \text{if } r < a_{\text{max}} \\ \frac{(M + kr^2)}{n} - r^2 & \text{if } r \geq a_{\text{max}}. \end{cases}$$

Then a $(0, k)$-length surgery that results in a sequence that satisfies the fundamental inequality is possible if and only if there exists $r_0$ such that $0 \leq h(r_0)$. This is equivalent to saying that the global maximum of $h$ is nonnegative. Because $h$ is a piecewise monotone function, the global maximum occurs at either $0, a_{\text{max}}$, or $\infty$. Note that $h(0) = \frac{M}{n} - a_{\text{max}}^2$, $h(a_{\text{max}}) = \frac{M}{n} + \left(\frac{k}{n} - 1\right) a_{\text{max}}^2$, and

$$\lim_{r \to \infty} h(r) = \lim_{r \to \infty} \frac{M + (k - n)r^2}{n} = \begin{cases} -\infty & \text{if } k < n \\ M/n & \text{if } k = n \\ \infty & \text{if } k > n. \end{cases}$$

Summarizing, when $k < n$ we have $\lim_{r \to \infty} h(r) < 0$ and also that $h(0) < 0$. So the desired $(0, k)$-length surgery is possible if and only if $h(a_{\text{max}}) \geq 0$, which is equivalent to $a_{\text{max}}^2 \leq \frac{M}{n-k}$. □

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