Stellar Atmospheres:
A Computational Approach

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Abstract

The stellar atmosphere contains the outermost layers of the star and includes the region from which photons escape the star and become visible. Particles in the stellar atmosphere are not completely ionized and the temperature, pressure and density are much lower than in the interior. Stars are so large and stellar atmospheres so thin, that the stellar atmosphere is often modeled as a slab. We develop three computer models to describe the characteristics of the stellar atmosphere. The first model calculates optical depth for a main sequence star using temperature, density and opacity at various radii. The second model compares the stellar atmosphere of a main sequence star with the stellar atmosphere of a red giant star. In the third model, we simulate a stellar atmosphere using the Monte Carlo method for radiative transfer which traces individual photons as they move through a model atmosphere. The behavior of each photon is driven by probabilistic interaction laws. In this type of model, a photon is emitted, travels a distance and is scattered or absorbed. An individual photon thus follows a “random walk” through the atmosphere, eventually exiting after a number of steps. We reproduce emergent intensity predicted by Chandrasekhar’s theoretical model for a semi-infinite atmosphere (1960). We also provide a visualization of one photon’s random walk through the model stellar atmosphere.

Background

Computational methods have been used by physicists for decades, but with the growing popularity and increased speed and performance of computers, the use of computers by physicists has grown tremendously. “Computational physics is now widely accepted as a third, equally valid complement to the traditional experimental and theoretical approaches to physics.”
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(DeVries 1994). Computational methods can be used to solve complex numerical problems which otherwise might not be solvable. Pang (2006) states that “computing has become a means of scientific study”. Frequently, a computational physicist uses numerical algorithms implemented on a computer in order to solve problems for which a mathematical model has been developed. The computer model is constructed specifically to the task of solving that particular problem.

Goals

The problem we seek to solve is to predict characteristics of the atmosphere of main sequence stars using one, or several, computer models. The questions we would like to address are: How should we define a stellar atmosphere? How do the composition and size of a star affect the size of the stellar atmosphere? How can we describe the process of radiative transfer from within the stellar atmosphere? From where within the star do photons emerge and at which angles are they most likely to emerge?

Method

The first task for a computational physicist is to determine a suitable environment for developing the computational model. In our case, the development environment needed to satisfy four criteria: the environment must (1) be computationally robust and include mathematical functions commonly used by physicists, (2) be easy to use and provide a stable and well-documented integrated development environment (IDE), (3) be affordable and accessible to students, and (4) be capable of running on several platforms favored by students including: Windows PC, Mac OS, and flavors of Linux. Our student/advisor team selected IDL data visualization and image analysis software from ITT Corporation as the preferred environment for
developing stellar atmosphere models because the software met each of the four criteria and because the advisor team had many years of experience using IDL.

In the end, three separate stellar atmosphere models were developed to answer questions posed by the research team. The simulations were created using IDL version 7.0 running on a laptop under Windows Vista (32 bit). In some cases, programs written in FORTRAN were used as a starting point for the simulations. Inputs were simplified and program comments were expanded from the original to make the IDL programs simple to use and understand. IDL plots were added and data files created for offline processing with spreadsheet applications. Structured programming techniques were used to make the IDL programs simple to maintain. IDL program listings are available online at the author’s home page (see Bibliography).

Astrophysical Concepts

Definition of Stellar Atmosphere

A star does not have a solid surface, so the definition of a stellar atmosphere is more difficult than for planetary bodies that do have solid surfaces. The atmosphere of a star is different than the interior of a star in several ways. Particles in the outer layers of the star are not completely ionized and the temperature, pressure and density are much lower than in the interior. The stellar atmosphere contains the outermost layers of the star and includes the region from which photons escape the star and become visible to us. Stars are so large and stellar atmospheres so thin that the stellar atmosphere is often modeled as a slab, called a plane-parallel stellar atmosphere. The effective temperature of the star is the temperature at which it would radiate if it were a blackbody radiator. The energy per second per unit area of a blackbody

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1 Intensity is a measure of energy radiated per second, so the luminosity per surface area of the star is the same as power per unit area of the blackbody radiator in the Stefan Boltzmann approximation.
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radiator follows the Stefan Boltzmann law (below). The temperature is equal to the effective
temperature somewhere within the stellar atmosphere.

\[
\frac{P}{A} = \sigma T^4
\]

The Stefan Boltzmann law where \( \sigma = 5.67 \text{E-8 watts/m}^2 \text{K}^4 \) and \( P/A \) is \( \text{j/m}^2 \text{s} \) in SI units.

**Optical Depth**

The motion of a photon through a gas follows a random walk because individual photons
are repeatedly scattered in random directions after encountering gas particles. Emission is any
process that adds photons to a beam of light and includes scattering photons into the beam as
well as emission of photons by electrons making downward transitions. Absorption is any
process that removes photons from a beam of light. Absorption includes scattering events as well
as absorption of photons by atomic electrons. A source function describes the ratio of emission to
absorption. The mean free path is the average distance a photon travels between collisions.

The change of intensity of a ray of light is proportional to its intensity, the opacity, the
distance traveled and the density of the gas. The distance is measured along the path of the beam.
Opacity, or the absorption coefficient, is the cross section for absorbing photons. The opacity of
a gas is a function of its composition, density and temperature (Carroll, Ostlie 1996). Intensity
and opacity are wavelength specific. Thus,

\[
dI_\lambda = -\kappa_\lambda \rho I_\lambda ds
\]

where \( \kappa \) is opacity, \( \rho \) is density and the minus sign indicates that the intensity decreases due to
the absorption of photons. For a beam of light traveling through a gas in the s-direction with
initial intensity \( I_{\lambda,0} \) at \( s = 0 \), the final intensity may be found by integrating. For a uniform gas of
constant opacity and density,

\[
I_\lambda = I_{\lambda,0} e^{-\kappa_\lambda \rho s} = I_{\lambda,0} e^{-\tau}
\]
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The optical depth, $\tau$, over a distance in a given direction is the number of photon mean free paths over that distance. Optical depth, $\tau$, is often used as the independent variable to describe the layers of the stellar atmosphere. Optical depth is more reliable than height as a measure of the stellar atmosphere because it more accurately describes the stellar material. The change in optical depth is related to opacity and density and is wavelength dependent. We assume $\tau = 0$ at the outermost layers of the star and that the light travels unimpeded after it is emitted from the star.

$$d\tau_\lambda = -\kappa_\lambda \rho ds$$

$$\tau_\lambda = \int_0^\infty \kappa_\lambda \rho ds$$

If $\tau_\lambda \gg 1$ for a light ray passing through a volume of gas, the gas is said to be optically thick, for $\tau_\lambda \ll 1$ the gas is optically thin.

A simple trapezoidal rule transforms the equation for optical depth and allows us to evaluate the integral (See Appendix A).

$$d\tau = -\kappa \rho ds$$

$$\tau_{i+1} - \tau_i = -\frac{(K_i \rho_i + K_{i+1} \rho_{i+1})(r_{i+1} - r_i)}{2}$$

Eddington Approximation

The Eddington approximation may be used to determine how temperature varies in a plane-parallel gray\(^2\) stellar atmosphere. The approximation was developed by English physicist Sir Arthur Eddington. For a stellar atmosphere in equilibrium, no net energy is added to or subtracted from the radiation field. The radiative flux is constant at every level in the atmosphere and is given by

$$F_{rad} = \sigma T_{eff}^4$$

\(^2\) A gray stellar atmosphere is one in which the opacity is assumed to be independent of wavelength.
where $T_{\text{eff}}$ is the effective temperature.

The intensity is assigned a value for the positive z-direction, $I_1$, and a value for the negative z-direction, $I_2$. There is no incident radiation at the outer boundary of the atmosphere, so $I_2(0) = 0$ at $\tau = 0$. Flux is a constant and the difference between $I_1$ and $I_2$ is constant throughout the atmosphere. The mean intensity, $\langle I \rangle$, is given by $\frac{1}{2} [I_1(\tau) + I_2(\tau)]$. We assume the stellar atmosphere is in local thermodynamic equilibrium so the source function is equal to the Planck function for a blackbody integrated over all wavelengths. Because the radiative flux is constant, the mean intensity is equal to the source function:

$$\langle I \rangle = S = B = \frac{\sigma T^4}{\pi}$$

The radiation pressure as a function of vertical optical depth is given by

$$P_{\text{rad}} = \frac{1}{c} F_{\text{rad}} \tau + C$$

where $F_{\text{rad}}$ is radiative flux, $c$ is the speed of light, $C$ is a constant of integration and we assume optical depth is constant over all wavelengths.

The radiation pressure may also be given as a function of mean intensity:

$$P_{\text{rad}} = \frac{4\pi}{3c} \langle I \rangle$$

Equating the two equations for radiation pressure and using boundary conditions to determine the constant of integration:

$$\frac{4\pi}{3c} \langle I \rangle = F_{\text{rad}} \left( \tau + \frac{2}{3} \right)$$

Lastly, we substitute the blackbody approximation for mean intensity, the approximation for radiative flux from above and rearrange to find the Milne-Eddington approximation for a gray atmosphere
The exact solution may be written as

\[ T^4 = \frac{3}{4} T_{\text{eff}}^4 \left( \tau + \frac{2}{3} \right) \]

where \( q(\tau) \) is the Hopf function that varies slowly with optical depth.

**Model 1: Optical Depth**

**Description**

The first simulation uses a simple trapezoidal rule to calculate optical depth at various points within a star using values of density and opacity of a model star. The temperature, density, and opacity of the star at various radii are obtained from STATSTAR, a one solar mass model of stellar structure (Carroll, Ostlie 1996). Several versions of the STATSTAR program are available, but all of them rely on similar assumptions: that the modeled stars are homogenous, main-sequence stars. The user is asked to define the stellar mass in solar units, the luminosity, the effective temperature, the mass fraction of hydrogen, and the mass fraction of metals. The program integrates from the surface of the star toward the center, stopping when a problem is detected or a solution is obtained. The pressure, density and temperature are all assumed to be zero at the surface of the star and the interior mass and luminosity are assumed constant through a number of surface zones. The program places a data file in the directory in which it was executed. Among other variables, the data file describes the temperature, density, opacity, and distance from the center for each layer of the star.

Our optical density program starts with the stellar model produced by STATSTAR (See Appendix B). Beginning at the outermost layer and working toward the center, the incremental
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optical depth is calculated using values from the current and prior layer using a trapezoidal estimation. The program keeps a running tally of optical depth as it calculates the incremental value for each layer.

Results

STATSTAR generated data for a one solar mass star with luminosity $1 \ L_{\odot}$ and radius $1 \ R_{\odot}$, mass fraction of hydrogen 0.70 and mass fraction of metals 0.008. The data from this model was used to calculate optical depth through the layers of the star. Figure 1 illustrates that optical depth increased dramatically with increasing temperature and depth.

![Graph illustrating optical depth, temperature, and radius for a main sequence star.](image)

Fig. 1. Comparison of optical depth with temperature and temperature with radius for a main sequence star. The graph on the left indicates how optical depth increases with temperature. The graph on the right illustrates that the highest temperatures are at the center of the star and that temperature decreases outward. Optical depth greater than 10 is considered opaque; hence, only the outermost layers of the star are transparent to light. Temperature and radius data provided by STATSTAR model.

In the STATSTAR model for this star, the effective temperature of 5500 K was reached in the first few layers of the star. An optical depth much greater than one is considered optically thick; our model illustrates that only the outermost layers of the star are transparent to light.
Model 2: Comparison of Stellar Atmospheres

Description

Given an understanding of optical depth, the next task was to develop a model to compare stellar atmospheres of two types of stars, a red giant star and a main sequence star. The model describes six physical parameters for each layer: temperature, gas pressure, radiation pressure, density, opacity and the effective acceleration of gravity. Each layer is also defined by optical depth and height (or depth from surface). The user is asked to specify the effective temperature, the surface gravitational acceleration and average mean molecular weight of the gas. The model steps through the atmosphere from the surface to the bottom using differential steps in optical depth, \(d\tau\). All other parameters are defined by the optical depth. The model also uses different step sizes based upon location within the atmosphere; for most steps, \(d\tau = 0.25 \times \tau\).

Parameters

Optical Depth (\(\tau\)): Optical depth increases inwards, but its zero level may be chosen at any location in the atmosphere. For our purposes, we define \(\tau = 0\) as the outer boundary of the atmosphere. We further simplify the model by stating that pressure, density and temperature also drop to zero at this boundary. The bottom of the atmosphere is defined as the layer where \(\tau = 1\) because the simplification for opacity used in the model does not hold up when \(\tau\) is too great. Also, our model will not handle convective layers near the bottom of the atmosphere. The surface of the star is defined where the optical depth is 2/3. The temperature will be equal to the effective temperature, \(T_{\text{eff}}\), at that position.

Temperature (\(T\)): The temperature is given as a function of the optical depth using the Eddington approximation:

\[
T^4 = \frac{3}{4} T_{\text{eff}}^4 (\tau + q(\tau))
\]
\[ q(\tau) = 0.7104 - 0.1331e^{-3.4488\tau} \]

From the top of the atmosphere, through the surface and to the bottom, the temperature ranges from 0.81\(T_{\text{eff}}\) to 1.06\(T_{\text{eff}}\), with \(T\) equal to the effective temperature at \(\tau = 2/3\).

**Gas Pressure (\(P_g\)) and Mean Molecular Weight (\(\mu\))**: The equation of state for an ideal gas relates gas pressure and molecular weight to temperature and density

\[ P_s = \frac{\rho RT}{\mu} \]

where \(R\) is the gas constant, \(R = 8.314E7\) erg/K mole in cgs units. The mean molecular weight depends upon the mixture of gases and their degree of ionization. This model uses data tabled from other more sophisticated models to determine mean molecular weight based on the log of the gravitational acceleration at the surface and the effective temperature. Two typical values of \(\mu\) are 1.048 for a main sequence star and 0.738 for a red giant.

**Radiation Pressure (\(P_r\))**: The radiation pressure arises from the absorption of photons. Photons transfer momentum when they are absorbed. A radiation field exists in the star with the flow of energy directed outwards. Particles are more likely to encounter a photon from the direction toward the center of the star; hence the force per unit area is more likely to push the particle outward. At radiative equilibrium

\[ P_r = \frac{1}{3}aT^4 \]

with \(a = 7.56464E-15\) erg/cm\(^3\)/K\(^4\)

Note that the radiation pressure decreases outward.

**Hydrostatic Equilibrium**: For a star at equilibrium, the gravitational field inward and the change in gas pressure outward are in balance. The general form of hydrostatic equilibrium, disregarding radiation pressure, is
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\[
\frac{dP}{dr} = -G \rho(r) \frac{M_r}{r^2}
\]

where \(P\) is the total pressure at distance \(r\) from the stellar center, \(\rho(r)\) is the density at that point, \(M_r\) is the amount of mass within a sphere with radius \(r\), \(G\) is the gravitational constant and \(r\) is the radial coordinate, sometimes called \(z\) in the atmosphere. In our model a form of the equation of hydrostatic equilibrium is used that corrects for radiation pressure by defining an effective gravitational acceleration, \(g_e\). The overall equation is cast as a function of \(d\tau\) and is solved numerically from the outside to the inside layer of the atmosphere.

\[
dP_{g_e} = \frac{g_e}{\kappa} d\tau
\]

In order to avoid a possible situation of dividing by zero, the initial value for density is chosen to be low, but not zero, at the outer layer of the star, \(\rho_0 = 1\times10^{-13} \text{ g cm}^{-3}\). Optical depth, \(\tau\), and height, \(z\), are set at zero at the outer layer. All other parameters are calculated from these starting points. Two examples are hard-coded in the program to make execution simpler, or the user may define the initial values for effective temperature, surface gravitational acceleration and average mean molecular weight when prompted. The first example is a main sequence star. The effective temperature is 10,000 K, surface gravitational acceleration is \(\log g_s = 4\) and average mean molecular weight, \(\mu = 1.048\). The user may select this star by entering “0” when prompted. The second example is a red supergiant star. In this case, the effective temperature is 10000K, surface gravitational acceleration is \(\log g_s = 2\) and a corresponding average mean molecular weight, \(\mu = 0.738\). The user selects this star by entering “1” when prompted.

**Results**

Our model calculated the atmosphere of the main sequence star at approximately 3400 km thick and the atmosphere of a red giant star at 500,000 km thick (Figure 2). The stars have
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the same effective temperature of 10,000 K and the effective temperature for both stellar models was achieved at optical depth \( \tau = 2/3 \), as expected.

![Fig. 2. Comparison of temperature with depth for a main sequence star (left) and a red giant star (right). Notice that the size of the stellar atmosphere of the red giant star is much larger than the stellar atmosphere of the main sequence star.](image)

We calculate a radius of 3.28 solar radii for the main sequence star and 80.38 solar radii for the red giant star.

We also compared optical depth for both of these stars (Figure 3). In each case, optical depth varied across the same range of values: 0 to 1, even though the depth of the atmospheres was dramatically different. This result illustrates that optical depth is a better parameter than physical distance for describing characteristics of a stellar atmosphere.

![Fig. 3. Comparison of optical depth with size of atmosphere for a main sequence star (left) and a red giant star (right). Notice that optical depth varies from zero to one for both atmospheres, although the atmospheres are very different in size. Optical depth is a better parameter than depth for describing stellar atmospheres.](image)
Model 3: Monte Carlo Simulation

Equation of Radiative Transfer

Traditionally, radiative transfer, the motion of photons through an atmosphere, is modeled by finding solutions to the equation of radiative transfer. We have seen that the intensity of a beam of light depends upon the rates at which photons enter and leave the beam (Astrophysical Concepts). Photons enter the beam through emission processes and leave the beam through absorption processes. The ratio of emission to absorption is called the source function, $S_\lambda$.

$$S_\lambda = \frac{j_\lambda}{\kappa_\lambda}$$

Here $j$ is the emission coefficient, $\kappa$ is the absorption coefficient and both coefficients are wavelength, or frequency, dependent. For an atmosphere in steady state (not changing with time), the equation of radiative transfer is:

$$-\frac{1}{\kappa_\lambda \rho} \frac{dI_\lambda}{ds} = I_\lambda - S_\lambda$$

In this case, $I_\lambda$ is the specific intensity, $S_\lambda$ is the source function, and $\rho$ is the density of the atmosphere. When the intensity of the light remains constant, $dI_\lambda/ds = 0$ and the intensity is equal to the source function (Carroll, Ostlie). Various solutions for the equation of radiative transfer are based on simplifying assumptions. For example, for a beam of light of initial intensity $I_{\lambda,0}$ entering a volume of gas of constant density, with constant opacity and a constant source function, the intensity of light as a function of the distance, $s$, traveled is

$$I_\lambda (s) = I_{\lambda,0} e^{-\kappa_\lambda \rho s} + S_\lambda (1 - e^{-s \kappa_\lambda \rho})$$

The equation of radiative transfer may be reformulated in terms of optical depth. Making the assumption that the atmosphere of the star is thin compared to the size of the star and assuming
the z-axis is the vertical direction, with \(z=0\) at the top of the atmosphere, the resulting assumption is of a plane slab, called a plane-parallel atmosphere. The general form of the transfer equation becomes

\[
\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda
\]

Integrating from the initial position of the ray inside the atmosphere to the top of the atmosphere, the resultant equation for the intensity at the top of the atmosphere is

\[
I_\lambda (0) = I_{\lambda,0} e^{-\tau_{\lambda,0}} - \int_{\tau_{\lambda,0}}^{0} S_\lambda e^{-\tau_\lambda} d\tau_\lambda
\]

Here the emergent intensity is equal to the initial intensity of the ray, reduced by absorption, and the emission (along the path from the initial optical depth to the surface), also reduced by absorption. The last term is on the right is positive because \(d\tau\) is negative; optical depth decreases in the direction of travel.

Solutions to the equation of radiative transfer using various simplifications were developed by S. Chandrasekhar (1960). Many solutions required numeric solution of the integrals involved.

**Monte Carlo Radiative Transfer**

Using Monte Carlo methods (Appendix C) to model radiative transfer is often simpler than solving the integrals required by the equation of radiative transfer, particularly for azimuthally symmetric geometries. The Monte Carlo method for radiative transfer traces individual photons through a model atmosphere. The behavior of each photon is driven by probabilistic interaction laws. In this type of model, a photon is emitted, travels a distance and is scattered or absorbed. Eventually the photon exits the atmosphere. Many features of the photon’s path are described by probability distribution functions. For example, the distance the photon
travels is described by a decaying exponential distribution. In this case, the random sample is weighted so that the underlying distribution function becomes an exponential distribution. Emission by point source, scattering angles and absorption rates all have underlying probability distribution functions and can be modeled using random numbers. The last step in a Monte Carlo model is to place the emerging photon into a bin based on the angles of exit. This process is repeated many times and the contribution of many photons in each bin leads to the overall intensity of the resultant light ray (Hoffman).

**Fundamental Principle**

The fundamental principle defines how we may sample points from a probability distribution function. To sample quantity $x_0$ from a probability distribution function $p(x)$ which is normalized over all $x$, we use the following:

$$\xi = \int_a^{x_0} p(x)dx$$

Where $\xi$ is a random number sampled uniformly from the range 0 to 1, and $a$ is the lower limit of the range over which $x$ is defined. In this case, $\xi$ is a uniform deviate. Uniform deviates are random numbers that lie within a specified range with any one number in the range just as likely as any other number. A reliable source of random deviates is essential to stochastic modeling (Press). Our model requires six distinct samples: one sample for optical depth, two samples for the angles of photon emission, two samples for the angles of photon scattering and one sample for absorption. Each sample is described by an underlying probability distribution. We weight the random numbers so that the resultant sample reflects the underlying probability distribution.

We begin the model with radiation from a point source. Radiation from a point source is radially uniform in all directions. Two angles are needed to define the direction in spherical coordinates, the azimuthal angle, $\varphi$, which can vary from 0 to $2\pi$ and the polar angle, $\theta$, which
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varies from 0 to $\pi$. In this model, we emit photons only in the upward direction from the base of the stellar atmosphere. We start by finding the cosine of the polar angle. Using the fundamental principle:

$$\xi = 2 \int_0^\mu \mu \, d\mu,$$

Where $\mu = \cos(\theta)$. This then gives:

$$\mu = \sqrt{\xi}$$

It is a simple matter to shift the random number from an equal distribution from 0 to 1 into an equal distribution along the range required for the azimuthal angle, $\varphi$.

$$\varphi = 2\pi \xi$$

Thus, for emission from a point source, two random numbers are generated and shifted to provide the angles required to define the direction the photon is moving through the atmosphere.

Another random number describes how far the photon travels through the atmosphere before scattering or absorbing. We use optical depth to calculate the distance traveled. The underlying probability distribution is described by optical depth as follows

$$p(\tau) \, d\tau = e^{-\tau} \, d\tau$$

Using the fundamental principle we find\(^3\)

$$\tau = -\log (\xi)$$

By sampling in this way, we adjust the random deviate for an exponential distribution. The distance, $L$, traveled by a photon (Whitney) is

$$L = \frac{\tau}{\tau_{\text{max}}}$$

And its position is converted to Cartesian coordinates using

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\(^3\) Note: we use log to refer to the natural logarithm. This is consistent with IDL and Mathematica™ programming. For example, in IDL ALOG returns the natural log and ALOG10 returns log base 10. In Mathematica, `Log[expr]` returns natural log and `Log[10, expr]` returns log base 10.
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\[ x = x + L \sin \theta \cos \varphi, \quad y = y + L \sin \theta \sin \varphi, \quad z = z + L \cos \theta \]

Finally, the photon interacts with the medium; it is either scattered or absorbed. To simplify the model, one can assume that all photons are scattered. Further, assume the scattering is isotropic, so a random sample of angles \( \varphi \) and \( \theta \) will determine the resultant angle. The angle \( \varphi \) is sampled over 0 to 2\( \pi \) and \( \mu \) is sampled over -1 to 1. Hence,

\[ \varphi = 2 \pi \zeta, \quad \mu = 2 \xi - 1 \]

**Emergent Photons**

Once a photon exits the slab, it is placed into a bin corresponding to its direction of travel. In the plane parallel slab atmosphere, each azimuthal angle \( \varphi \) is equally probable, so the photons are binned in \( \mu \). Also, only the photons emerging from the top of the slab are of interest, so photon are binned in the range 0 < \( \mu < 1 \) or \( \theta \) between 0 degrees and 90 degrees. We may compare results from this model with results obtained by Chandresekhar for a similar atmosphere (Whitney).

In a plane parallel slab, intensity moments are calculated with respect to optical depth through the slab. The slab is split into layers of equal depth and the photons are tallied, weighted by powers of their direction cosines, to obtain three intensity moments: \( J \) is the mean intensity, \( H \) is the flux and \( K \) is the radiation pressure. In our model, the three intensity moments are calculated as follows:

\[ J = \frac{B_\nu}{4 N_0} \sum \frac{1}{|\mu_i|}, \quad H = \frac{B_\nu}{4 N_0} \sum \frac{\mu_i}{|\mu_i|}, \quad K = \frac{B_\nu}{4 N_0} \sum \frac{\mu_i^2}{|\mu_i|} \]

**Results**

The computer simulation processed one million photons through the model atmosphere, tallied the moments and produced a graph of the intensity moments with optical depth (Figure 4). In this atmosphere, the maximum optical depth is ten at the bottom of the atmosphere and one in
the outermost layer of the atmosphere. We expect net flux, the H moment, to be constant through the layers of the atmosphere in this simulation because each photon that enters a layer eventually leaves the layer; there is no absorption or emission.

A radiation pressure gradient exists in the model atmosphere, indicated by the decreasing K intensity moment in the graph above. A radiation pressure gradient produces the gradual movement of photons from the interior to the outer layers of a star. Each photon follows a random walk, but there is a slight net motion toward the outer layer of the atmosphere. The graph illustrates that the Monte Carlo simulation reproduces the gradient predicted by radiative transfer theory (a hopeful sign).

The exit angle of each emergent photon is calculated and the photon is binned into one of ten bins according to angle. After one million photons are processed, the results are normalized and plotted. Figure 5 is the emergent intensity by exit angle. Most photons emerge from the atmosphere with small angles (from the normal). Fewer photons emerge at larger angles. These results agree with numerical estimates made by S. Chandrasekhar for a plane parallel, semi-infinite atmosphere with isotropic scattering (1960). The results of the Monte Carlo simulation (diamonds) are plotted along with Chandrasekhar’s estimates (curved line) in Figure 5.
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A single photon moving through the model atmosphere should exhibit a random walk. We created a program to follow the progress of a single photon through the Monte Carlo model atmosphere. A plot is updated and displayed with each random step. After execution, a 3D plot is displayed that may be rotated along all three axis. Figure 6 illustrates two examples of a random walk taken by a single “photon”.

The model atmosphere is one unit thick in the z- direction and infinite in the x- and y- directions. The bottom of the model stellar atmosphere is z = 0 and z = 1 is the outer edge. The simulation ends when the photon is emitted.
Conclusions and Next Steps

The simulations presented here are an excellent first step at using computational physics to explore properties of stellar atmospheres, but there are many improvements that may be made to the models. Absorption and reemission of photons within the stellar atmosphere are two fundamental properties that are not addressed by the models. In addition, the models assume that scattering is isotropic. There are many mechanisms for scattering that are not addressed here, including scattering by dust grains and Thomson scattering. The models do not account for polarization of emitted light. They also focus primarily on main sequence stars and could be expanded to include other types of stars. Further development will address these issues as well as a modification to replace the slab atmosphere with an azimuthally symmetric model.
Bibliography

Appendix A

Numeric Integration Techniques

One of the most common tasks in computational physics is to develop a numeric solution to a problem that does not lend itself to analytic solution. Numeric integration is a fundamental tool used by computational physicists and integration by quadrature is the simplest technique. Integration by quadrature involves finding the edge length of a four-sided regular polygon with an area equal to that bounded by the area of the original curve. The technique is similar to “squaring the circle”, or the process of finding a square whose area is exactly equal to the area of a circle. The most common approach to quadrature is to use the trapezoid rule. The region under a curve is divided into \( N \) segments of equal width. The area of each resulting trapezoid is approximated by taking the average value of the height multiplied by the width. The areas of the \( N \) trapezoids are added to give the approximate value of the integral. The resulting fit is piecewise linear (DeVries 1994)

\[
\int_{x_0}^{x_N} f(x)dx \approx \frac{h}{2}(f_0 + f_1) + \frac{h}{2}(f_1 + f_2) + \ldots + \frac{h}{2}(f_{N-1} + f_N)
\]

\[
= h\left(\frac{f_0}{2} + f_1 + f_2 + \ldots + f_{N-1} + \frac{f_N}{2}\right)
\]

A quadratic approximation may be found using a composite Simpson’s rule

\[
\int_{x_0}^{x_N} f(x)dx \approx \frac{h}{3}(f_0 + 4f_1 + 2f_2 + \ldots + 2f_{N-2} + 4f_{N-1} + f_N)
\]

DeVries derives the Euler-Maclaurin integration formula (1994) which is the piecewise linear trapezoid rule, above, with correction terms. DeVries notes that the Euler-Maclaurin formula is superior to other numeric integration techniques, especially when the derivatives vanish at the limits of integration.
\[
\int_{x_0}^{x_N} f(x)\,dx = h\left(\frac{f_0}{2} + f_1 + f_2 + \ldots + f_{N-1} + f_N\right) + \frac{h^2}{12}\left[f_0' - f_N'\right] - \frac{h^4}{720}\left[f_0'' - f_N''\right] + \ldots
\]

IDL provides a library of numeric integration functions of varying complexity. The functions QSIMP and QROMB evaluate integrals over a closed interval and are based on routines described in *Numerical Recipes in C: The Art of Scientific Computing* (Second Edition). In that text, Press, Teukolsky, Vettering and Flannery recommend Romberg integration, which uses a variety of different orders, as a method of choice. In IDL, QSIMP is numerical integration based on Simpson’s rule and QROMB is Romberg integration.
Appendix B

Sample STATSTAR Listing (inner and outer layers only)

A Homogeneous Main-Sequence Model

The surface conditions are:
The central conditions are:

- \( M_{\text{tot}} = 1.000000 \text{ Msun} \)
- \( R_{\text{tot}} = 1.100596 \text{ Rsun} \)
- \( L_{\text{tot}} = 1.000000 \text{ Lsun} \)
- \( T_{\text{eff}} = 5500.000000 \text{ K} \)
- \( X = 0.700000 \text{ } \)
- \( Y = 0.290000 \text{ } \)
- \( Z = 0.010000 \text{ } \)
- \( Q_m = 1.0 - M_r/M_{\text{tot}}, \text{ where } M_{\text{tot}} = 1.989000 \times 10^{33} \text{ g} \)
- \( Q_mL_r T \rho \kappa \epsilon \text{ may be limited to } +99.9 \text{ or } -99.9; \text{ if so it is labeled by } * \)

<table>
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<th>( r )</th>
<th>( Q_m )</th>
<th>( L_r )</th>
<th>( T )</th>
<th>( P )</th>
<th>( \rho )</th>
<th>( \kappa )</th>
<th>( \epsilon )</th>
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Appendix C

The Monte Carlo Method

The Monte Carlo method may be used for stochastic modeling. Stochastic models estimate possible outcomes using random events; they differ from deterministic algorithms by accounting for a range of possible outcomes. The Monte Carlo method is a useful technique when an underlying process is random or when its probability distribution function is known. A function is evaluated at a random sample of points, and the integral is estimated based on that random sample. The Monte Carlo method may be used to calculate the area of an irregular shape, Figure 1, for example.

![Area Calculation](image)

Assume that the distribution of sampling points is random and uniformly scattered, and that the number of points sampled is \( N \). The number of sampling points that fall within the shape, \( S \), is \( N' \). The area of \( S \) is approximately equal to the ratio \( N'/N \) (Sobol). In Figure 3, 19 of 50 points fall within \( S \); the calculated area is 0.38. The known area is 0.36.
Stellar Atmospheres: A Computational Approach

The motion of a photon through a layer of a star is approximately a random walk. The direction that the photon takes after a collision may be modeled by sampling a random number; in this case the random number represents the cosine of the angle of scattering.

There are three advantages to the Monte Carlo method. First, the algorithm has a very simple structure. The computer program selects one random point at a time, and the trial is repeated many times. Each trial is independent of every other and the results are averaged. Second, the error is proportional to \( \sqrt{D/N} \) where D is a constant and N is the number of trials. In order to reduce the error by a factor of 10, one would increase the number of trials by 100. Several Monte Carlo methods exist to reduce the value of D to improve accuracy. Third, problems that are not intrinsically random can be connected with an artificially constructed probabilistic model; hence the Monte Carlo method may be used to solve a wide variety of problems (Sobol). The primary disadvantage of the Monte Carlo method is that it can take a long time to calculate the number of points necessary to achieve adequate resolution. Additionally, it is essential that a method exist for determining whether a random point is inside or outside the region of integration (Press).