How can a photon, with no mass, have momentum?

Short answer:
The momentum of a photon is defined in special relativity. You have to use relativity to calculate the momentum of a photon.

It's a bit like how understanding "a photon can act like a particle", and "particles can act like waves" requires the beginnings of quantum physics.

Classically, we consider only the kinetic energy of motion or the potential energy. For example, a rock held high, when dropped in a gravitational force, converts potential energy to kinetic energy. There is the mass energy ($E=mc^2$) of the rock, but it does not change while the rock is falling so it is a constant that we ignore. In special relativity, however, we use not only the kinetic energy and potential energy but also the mass energy when considering conservation of energy. This is especially important when considering the familiar equation $E = mc^2$ where we will see a high energy photon converting some of its energy into an electron positron pair. The mass of the two electrons having come from the energy of the photon. Likewise, the classical definition must be revised to take relativistic effects into account.

When the relativistic and mass considerations are taken into account, it turns out that the energy of a photon is given by $h\nu$ and its momentum by $h\nu/c$.

For those who can do a little math:

Classically (i.e. before relativity), momentum, $p = m_o v$.

Relativistically, energy and momentum are combined in the following equation.

$$E^2 = (pc)^2 + (m_o c^2)^2$$

where m_o represents the rest mass, the mass of a particle at zero velocity.
In relativity, m=m(v), which means mass is a function of the velocity of the particle.
When v is near the velocity of light, c, there are what we call "relativistic corrections".

Relativistically, momentum is given by p=mv, but m is not constant.
$m = m_o \, \mathrm{sqrt}(1-(v/c)^2)$, where m_o is the energy of a particle at rest.

So, momentum is given by $p = m_o v/\mathrm{sqrt}(1-(v/c)^2)$. Note that when $v << c$, this reduces to the classical form $p = m_o v$.

What is the momentum of a photon? Recall $E^2 = (pc)^2 + (m_o c^2)^2$

Now $m_o = 0$, so $E = pc$ and the above equation reduces to $p = E/c$. Since the energy of a photon is Planck's constant h times the frequency of the photon, the momentum is $p = h\nu/c$.